Notes on approximations for turbulent flows with large temperature differences

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(i) The diffusion terms in the mean velocity and temperature equations of turbulent flow are analysed to decide when variations of fluid properties can produce appreciable errors.

(ii) A theoretical demonstration is given that in the mean-flow continuity equation for a gas the error in assuming constant density is small if the flow is turbulent, even when the temperature variations are large.

(iii) Separate discussion is given of the case of local heat sources in turbulence, as large errors can occur there.

1. Introduction

In theoretical work on low-speed flows with temperature variations it is common to suppose that all temperature differences are small enough for changes in fluid properties to be neglected. In experimental work the supposition is frequently not fulfilled. In gas flows, in particular, temperature differences may well be insufficiently small compared with the absolute temperature for density variations to have no effect. In any fluid, variations of viscosity and thermal conductivity may be sizable. These notes consider, for turbulent flows, some aspects of the errors so caused. They do not give corrections—the complexity of the accurate equations precludes that—but discuss the magnitude of the errors for cases in which some analysis is necessary before this can be properly estimated. It might be thought, for instance, that for mean flows, where the direct action of molecular effects is negligible, temperature variation of viscosity would not matter, but this is seen in § 2 to be fallacious. Then, in §§ 3 and 4, attention is drawn to a respect in which theory fails less than would appear at first sight and also to a situation in which there is particular danger of large errors.

The notation is standard. An overbar ($^-$) indicates a mean value and ($'$) the difference between the instantaneous value and the mean value.

The estimation of errors is just a matter of writing the relative sizes of retained and neglected terms in terms of temperature, velocity, and length scales. During this procedure, it must be noted that there are two temperature scales (and, correspondingly, two density scales); the scale of $\overline{T}$ is that of the absolute temperature T_0, whereas that of T', or of $\overline{T}$ when it appears as a gradient, is ΔT, the difference between extremes of temperature. Strictly, a further distinction should be made between these last two and, similarly, between the scales of mean and fluctuating velocities, but this is left out of account; otherwise it is necessary to present the analysis somewhat differently for different flows; the distinctions are unimportant in the end. The use of a single velocity scale, U, further does not allow for flows such as a wake in which the mean velocity difference has a smaller scale than the mean velocity, but it can be shown that the situations in which the results are different are of no physical importance. For length scales we distinguish between longitudinal gradients having scale L and transverse gradients having scale δ, as the results are of interest mainly for boundary-layer-type flows. (These apply, of course, to gradients of mean quantities; we shall be meeting a turbulence length scale in § 2.)

2. Mean velocity and temperature equations with variable fluid properties

Consider the term

$$A = \overline{\frac{\partial}{\partial x_i}\left(\mu \frac{\partial u_j}{\partial x_i}\right)}$$

in the mean velocity equation. If μ varies with temperature,

$$A = \underbrace{\overline{\mu \frac{\partial^2 u_j}{\partial x_i^2}}}_{B} + \underbrace{\overline{\frac{d\mu}{dT}\frac{\partial T}{\partial x_i}\frac{\partial u_j}{\partial x_i}}}_{C}.$$

C will behave, at least so far as the subsequent order of magnitude analysis is concerned, similarly to

$$\alpha \frac{\mu}{T}\overline{\frac{\partial T}{\partial x_i}\frac{\partial u_j}{\partial x_i}}, \quad \text{where} \quad \alpha = \frac{T}{\mu}\frac{d\mu}{dT} \text{ (a function of } T).$$

Now

$$\overline{\frac{\partial T}{\partial x_i}\frac{\partial u_j}{\partial x_i}} = \frac{\partial \overline{T}}{\partial x_i}\frac{\partial \overline{u}_j}{\partial x_i} + \overline{\frac{\partial T'}{\partial x_i}\frac{\partial u_j'}{\partial x_i}},$$

and the length scale of the second part is a turbulent, not a mean flow, one. Consequently A can be much larger when μ is variable than when it is constant. In fact $C \simeq \alpha\mu\Delta T U/T_0\lambda^2$, where λ is the dissipation-length parameter. [There are two assumptions involved in this stage. First, no distinction is being made between the kinetic energy and $\overline{T'^2}$ dissipation-length parameters. Hence the argument as it stands does not cover the case of Prandtl number very different from 1, though it could be modified to do so (nor is the configuration considered in § 4 covered by this). Secondly, it is assumed that the correlation

$$R = \overline{\frac{\partial T'}{\partial x_i}\frac{\partial u_j'}{\partial x_i}} \Big/ \left\{\overline{\left(\frac{\partial T'}{\partial x_i}\right)^2}\,\overline{\left(\frac{\partial u_j'}{\partial x_i}\right)^2}\right\}^{\frac{1}{2}}$$

(summation convention not applying here) is appreciable for at least some components. The condition for it to be so may be inferred from the equation

$$\frac{\partial}{\partial t}(\overline{u_j'T'}) + \overline{u_i'u_j'}\frac{\partial \overline{T}}{\partial x_i} + \overline{u_i'T'}\frac{\partial \overline{u}_j}{\partial x_i} + \overline{u}_i\frac{\partial}{\partial x_i}(\overline{u_j'T'}) + \frac{\partial}{\partial x_i}(\overline{u_i'u_j'T'}) + \frac{1}{\rho}\overline{T'\frac{\partial p'}{\partial x_j}}$$
$$- \kappa\overline{u_j'\frac{\partial^2 T'}{\partial x_i^2}} - \nu\overline{T'\frac{\partial^2 u_j'}{\partial x_i^2}} \text{ (+ further terms if buoyancy forces are significant)} = 0. \quad (1)$$

(That this is the constant fluid property form does not matter for the present considerations.) This represents the balance of a component of the temperature flux (see Ellison 1957, § 6 and, in particular, equations (16) and (19) for an example of this). The seventh and eighth terms indicate molecular effects on this balance; they are at least partly dissipative, the second, third, and perhaps the buoyancy force terms being the corresponding productive ones. Now these two molecular terms are closely related to the correlation under consideration, at least when the Prandtl number is near 1, because

$$\overline{u_j'\frac{\partial^2 T'}{\partial x_i^2}}+\overline{T'\frac{\partial^2 u_j'}{\partial x_i^2}}=\frac{\partial^2}{\partial x_i^2}(\overline{u_j'T'})-2\overline{\frac{\partial T'}{\partial x_i}\frac{\partial u_j'}{\partial x_i}}.$$

It may be inferred that R is appreciable when the balance of $\overline{u_j'T'}$ plays a role in the turbulence comparable with those of kinetic energy and $\overline{T'^2}$. Thus it is appreciable in shear flows with heat transfer but not in grid turbulence with uniform mean temperature.]

Continuing the estimation of the magnitude of C, a little manipulation of the various scales in turbulence (Hinze 1959, pp. 185–6), or a consideration of the energy balance in a shear flow, gives $\lambda^2 \simeq \delta\nu/U$. Hence

$$C \simeq \frac{\alpha\rho_0 \Delta T U^2}{T_0 \delta}.$$

We compare this with the magnitude of another term in the same equation, $\rho[\partial(\overline{u_i'u_j'})/\partial x_i]$ (D, say): $D \simeq \rho_0 U^2/\delta$; hence $C/D \simeq \alpha\Delta T/T_0$. Thus for large temperature variations in a fluid having $\alpha \simeq 1$, it is incorrect to suppose that molecular terms in a mean equation for turbulent flow are small to the order ν/ν_T (where ν_T is the eddy viscosity), as is the case when fluid properties are constant.

The reader may well be wondering why this discussion has been given in terms of variable viscosity, when it would seem that variation of thermal conductivity is a simpler case; for

$$(E=)\overline{\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)}=\overline{k\frac{\partial^2 T}{\partial x_i^2}}+\overline{\frac{dk}{dT}\left(\frac{\partial T}{\partial x_i}\right)^2}, \tag{2}$$

and the second term is analogous to the one considered above without the complication of the correlation. However, we shall now see that the same effect does not occur here. We may make the transformation

$$\overline{\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)}=k_0\frac{\partial^2}{\partial x_i^2}\left(\overline{\int^T \frac{k}{k_0}dT}\right), \tag{3}$$

(k_0 is the value of k at the reference temperature T_0), showing that a term involving turbulence length scales need not be brought in and that E is of the same order of magnitude as when k is constant.

This conclusion is disguised in (2) by the correlation of k-fluctuations with T-fluctuations making $\overline{k(\partial^2 T/\partial x_i^2)}$ also large in a way that just cancels the effect of $\overline{(dk/dT)\,(\partial T/\partial x_i)^2}$. This in turn suggests that, if any other temperature-dependent quantity is a coefficient of $\partial^2 T/\partial x_i^2$, care is again needed. Consider in this connexion the full mean temperature equation

$$\bar{u}_i \frac{\partial \bar{T}}{\partial x_i} + \overline{u_i' \frac{\partial T'}{\partial x_i}} = \overline{\frac{1}{\rho C_p} \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right)}. \tag{4}$$

We may anticipate from the above that, although the temperature variation of k does not produce appreciable terms on the right-hand side, that of ρ does. It is seen that this is indeed the case by making the transformation

$$\overline{\frac{1}{\rho C_p} \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right)} = \kappa_0 \frac{\partial^2}{\partial x_i^2}\left(\overline{\int^T \frac{\kappa}{\kappa_0} dT}\right) - \overline{k \frac{d(1/\rho C_p)}{dT}\left(\frac{\partial T}{\partial x_i}\right)^2}, \quad \left(\kappa = \frac{k}{\rho C_p}\right).$$

Analysis as before shows that the second part of this is small compared with the left-hand side of (4) only to the order $\Delta T/T_0$ when $\beta = -\rho C_p T[d(1/\rho C_p)/dT]$ is of order 1 ($\beta \equiv 1$ for a perfect gas).

It is probably misleading to regard this last result as an occasion when molecular effects are important in turbulent flow, for the same error in the usual approximation may be detected by retaining ρC_p on the left-hand side of the equation, thus:

$$\overline{\rho C_p u_i \frac{\partial T}{\partial x_i}} = \overline{\frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right)}. \tag{5}$$

There is then an approximation of order $\Delta\rho/\rho_0$ in taking the left-hand side of (5) as $\rho_0 C_p \bar{u}_i(\partial \bar{T}/\partial x_i) + \rho_0 C_p \overline{u_i'(\partial T'/\partial x_i)}$, but very little approximation in dropping the right-hand side. The important point is just that it is fallacious to argue that the equation can be put in the form (4) and the right-hand side then dropped because it is a 'molecular term'.

We now return to the velocity equation. The above comments about ρ-variability apply again. But we have already seen that μ-variability also has an effect; there is no analogue to (3). Intuitively it seems probable that μ-variability will not come in if the boundary conditions on velocity and temperature are similar and if, further, $k(T)$ and $\mu(T)$ have similar functional forms. In general, however, it does come in. It is particularly important to remember this for fluids for which the variation of μ is more marked than that of ρ (α large compared with β); water is an important example in which this is strongly so. Then there may be an approximation in the velocity equation but little in the temperature equation.

All the preceding discussion has been concerned with the approximation in

$$\bar{u}_i \frac{\partial \bar{u}_j}{\partial x_i} + \overline{u_i' \frac{\partial u_j'}{\partial x_i}} = -\overline{\frac{1}{\rho} \frac{\partial p}{\partial x_j}} + \nu \frac{\partial^2 \bar{u}_j}{\partial x_i^2}$$

and

$$\bar{u}_i \frac{\partial \bar{T}}{\partial x_i} + \overline{u_i' \frac{\partial T'}{\partial x_i}} = \kappa \frac{\partial^2 \bar{T}}{\partial x_i^2}.$$

There are further approximations of the order $\Delta\rho/\rho_0$ in the common practice of replacing the second terms by $\partial(\overline{u_i' u_j'})/\partial x_i$ and $\partial(\overline{u_i' T'})/\partial x$.

3. Continuity equation of mean flow

The continuity equation of mean flow is usually taken in the form

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0.$$

It will now be shown that this is a good approximation in turbulent flow of gases even when the temperature variations are large—far better than is immediately apparent. Physically this means that expansion effects shift the mean flow streamlines less than in laminar flow (for which $\partial u_i/\partial x_i = 0$ is quite a poor approximation). The result is probably most useful for the case of a boundary layer on a heated wall—the temperature difference can remain large at all distances downstream—but the treatment here will be general.

The accurate continuity equation may be written

$$\frac{\partial u_i}{\partial x_i} + \frac{1}{\rho}\frac{D\rho}{Dt} = 0. \tag{6}$$

The second term is similar to the left-hand side of the temperature equation

$$\rho C_p \frac{DT}{Dt} = \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right), \tag{7}$$

and may thus be replaced by one similar to the right-hand side. This is a molecular conductivity term and the result of this section derives in essence from the unimportance of molecular effects in turbulent flow (it will be seen that the exception to this described in § 2 does not usually arise here). Substitution of $\rho T = \rho_0 T_0$ in (7) gives

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{1}{\rho_0 T_0 C_p}\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)$$

and so from (6)

$$\frac{\partial u_i}{\partial x_i} = \frac{1}{\rho_0 T_0 C_p}\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) = \frac{1}{\rho_0 T_0 C_p}\frac{\partial^2}{\partial x_i^2}\left(\int^T k\,dT\right) \tag{8}$$

$$= \frac{\kappa_0}{T_0}\frac{\partial^2}{\partial x_i^2}\left(\int^T \frac{k}{k_0}dT\right) \quad (C_p \text{ independent of } T).$$

We then obtain, on taking average values,

$$\underset{F}{\frac{\partial \bar{u}_i}{\partial x_i}} = \underset{G}{\frac{\kappa_0}{T_0}\frac{\partial^2}{\partial x_i^2}\left(\overline{\int^T \frac{k}{k_0}dT}\right)}. \tag{9}$$

We now consider the order of magnitude of either side of (9):

$$F \simeq U/L; \quad G \simeq \kappa\Delta T/T_0\delta^2.$$

(It is possible to think of a form of $k(T)$ for which the use of ΔT as the scale of the difference of $\overline{\int^T \frac{k}{k_0}dT}$ is unsatisfactory, but such is most unlikely to apply to a real fluid. This is, for instance, the correct scale for any power law.) Thus we have

$G/F \simeq \Delta T \kappa L / T_0 U \delta^2$. But $L^2/\delta^2 \simeq UL/\nu_T$. (It is because this increase in the boundary-layer thickness through an eddy viscosity replacing a molecular one is essential to the argument that it does not also apply to laminar flow.) Therefore,

$$\frac{G}{F} \simeq \frac{\Delta T}{T_0} \frac{\nu}{Pr\,\nu_T},$$

which will be small even if $\Delta T/T_0$ is not. The fractional error in putting $F = 0$, i.e. $\partial \bar{u}_i/\partial x_i = 0$, will be correspondingly small.

The preceding analysis made use of the perfect-gas equation. This should apply in most cases of practical importance. However, brief mention may be made of what happens if a different equation of state applies; the situation is somewhat more complicated, and it may clarify the nature of the analysis to see why. For any other equation of state, the equation corresponding to (8) has temperature-dependent quantities in the coefficient on the right-hand side. Consequently averaging cannot be carried through the differentiation and, in the same way as in § 2, a term involving turbulent length scales comes in. The magnitude of this may be estimated as before;† it is of order $(\Delta T/T_0)^2$ smaller than F, which still compares favourably with a fractional error of $\Delta T/T_0$ in laminar flow. The still smaller error in the perfect-gas case results from a coincidental cancelling of two non-linearities, that in the equation of state and that due to the temperature dependence of ρ in the term $\rho C_p(DT/Dt)$ of the temperature equation.

Expansion effects may remain important in a laminar sublayer; indeed the above argument suggests that, in view of the large gradients, they may be particularly serious there even for comparatively small $\Delta T/T_0$. However, qualitative prediction of their behaviour might be possible by analogy with flows with injection through the wall.

4. Temperature difference introduced locally into pre-existing turbulence

So far it has been supposed that the length scales of the mean velocity and temperature fields are the same. This is not so when, as in turbulent-diffusion experiments, the heat source is a small element placed in an already turbulent flow field. A full analysis for this case of all the aspects considered above is complicated and, so far as I can see, not very rewarding. However, this is a situation in which particularly large errors can arise. We therefore consider one example of the way this can happen, by examining the relative sizes of $\overline{u_j'(\partial u_i'/\partial x_i)}$ (H, say) and $\partial(\overline{u_i'u_j'})/\partial x_i$ (J, say), the former being the term neglected in the usual practice of taking the latter as the gradient of the stress. If the one is not small compared with the other, the supposition that the velocity field is unchanged by the introduction of the heat source is without justification (although the ratio H/J may overestimate the change; as the temperature difference is increased, the flow might adjust itself so that the gradient of the true stress remains constant rather than the gradient of $\overline{u_i'u_j'}$).

† Though this is a point at which the distinction between the scales of mean and fluctuating quantities (see § 1) *is* necessary to the argument.

This particular example was chosen primarily because it is probably the most serious one, but also because it illustrates the way in which the unapproximated continuity equation (6), with the division into mean and fluctuating parts applied to both velocity and density, can be used for estimating the magnitude of such effects. Equation (6) may be written

$$\frac{\partial u_i}{\partial x_i} = -\frac{\partial r}{\partial t} - u_i \frac{\partial r}{\partial x_i},$$

where $r = \log \rho/\rho_0$ (and thus has $\Delta\rho/\rho_0$ as its scale and a spatial distribution similar to that of temperature). Subtraction of the average of this equation from the unaveraged form gives

$$\frac{\partial u_i'}{\partial x_i} = -\frac{\partial r'}{\partial t} - u_i' \frac{\partial \bar{r}}{\partial x_i} - \bar{u}_i \frac{\partial r'}{\partial x_i} - u_i' \frac{\partial r'}{\partial x_i} + \overline{u_i' \frac{\partial r'}{\partial x_i}}$$

and so

$$\underset{H}{\overline{u_j' \frac{\partial u_i'}{\partial x_i}}} = -\underset{K}{\overline{u_j' \frac{\partial r'}{\partial t}}} - \underset{L}{\overline{u_i' u_j'} \frac{\partial \bar{r}}{\partial x_i}} - \underset{M}{\bar{u}_i \overline{u_j' \frac{\partial r'}{\partial x_i}}} - \underset{N}{\overline{u_i' u_j' \frac{\partial r'}{\partial x_i}}}. \tag{10}$$

This can be used for estimating the size of H. The difference from previous cases is brought about by the heat wake having different length scales from the main flow; we will call its width δ_H. From (10) we get $H \simeq \Delta\rho U^2/\rho_0 \delta_H$. This is immediately indicated by the term L.† Furthermore, $J \simeq U^2/\delta$, and so

$$\frac{H}{J} \sim \frac{\Delta\rho}{\rho_0} \frac{\delta}{\delta_H},$$

which is not necessarily small. As an illustration of the possible effect in an experimental set-up, consider δ/δ_H in the region of measurement in Corrsin & Uberoi's (1951) work on local heat sources. It is about 10. These authors are not explicit about the magnitude of $\Delta\rho/\rho_0$, but it seems likely from the temperature and size of the source that it was somewhere between 0·02 and 0·1. Thus we see that H/J can be quite appreciable.

A similar analysis may be carried through for the effect of non-zero $\partial u_i'/\partial x_i$ on the use of $\partial(\overline{u_i'T'})/\partial x_i$ in the temperature equation. However, this is as usual small to the order $\Delta\rho/\rho_0$; δ_H is the length scale in all the terms. Diffusion of heat from the source is seriously affected only through changes in the mean velocity field.

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† The other terms are not so readily estimated, but there is no reason to expect either that they are any larger or that they just cancel L so as to make H smaller. In point of fact, arguments invoking the similarity of these terms to ones appearing in equation (1) suggest that K, M and N are likely to be of the same order as L.